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Abstract

The aim of this paper is two-fold: first, to emphasize that the seminal result of Dow and Werlang [7] remains valid under weaker conditions, and this even if non-positive prices are considered, or equally that the no-trade interval result is robust when considering assets which can yield non-positive outcomes. Second, to make precise the weak uncertainty aversion behavior characteristic of the existence of such an interval.

Keywords: Choquet expected utility, no-trade interval, perfect hedging, comonotone diversification, capacity.

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1 Introduction

In a seminal paper [7], Dow and Werlang dealt with the basic portfolio problem under uncertainty. They proved that for an uncertainty averse Choquet expected utility decision maker (CEU DM), endowed with a convex capacity v and a C^2 concave increasing utility function u , there exists an interval of prices $[I(X), -I(-X)]$ within which this agent neither buys nor sells short some shares of an asset X . The highest price at which the agent will buy the asset is the expected value of the asset under v (i.e. the Choquet integral of X , $I(X)$) whereas the lowest price at which the DM sells the asset is the expected value of selling the asset short (i.e. $-I(-X)$). Such a behavior is intuitively plausible and compatible with observed investment behavior. It contrasts with the prediction of expected utility theory under risk (see von Neumann and Morgenstern [16]), according to which a strongly risk averse agent (i.e. with a concave utility function) or equivalently a weakly risk averse agent (a result due to Rothschild and Stiglitz [13]) will invest in an asset X if and only if the expected value of this asset exceeds the price, and will wish to sell the asset short if and only if the expected value is lower than its price and consequently will have no position in the asset if and only if the price is exactly $E(X)$ (see Arrow [4]).

In "Portfolio inertia under ambiguity" [5], Asano studied the portfolio inertia phenomenon in the context of ambiguity. His model consists of an investor who is a CEU maximizer and a quadruple (S, \mathcal{A}, p, u) where S is a set, \mathcal{A} a λ -system, p a probability on \mathcal{A} and u a concave increasing function (the utility function of the DM). The beliefs of the investor are captured through the inner measure $p_{\mathcal{A}}$ defined for all $B \in 2^S$, by $p_{\mathcal{A}}(B) = \sup \{p(A) \mid A \in \mathcal{A}, A \subset B\}$. In this context, he shows that the no-trade interval phenomenon persists. More precisely, he shows that $(\int X(s)p_{\mathcal{A}}(ds), -\int (-X(s))p_{\mathcal{A}}(ds))$ is a no-trade interval. It is to be noted that the inner measure $p_{\mathcal{A}}$, while still super-additive¹, is not necessarily convex. In fact, Asano shows that all super-additive capacities exhibit portfolio inertia and therefore that convexity is not necessary.

In the present paper, we show that super-additivity at certainty² is the weakest possible condition for the existence of portfolio inertia. More precisely, we show that for a CEU investor whose beliefs are captured through a capacity v and whose preferences are given by a C^1 increasing utility function u , the existence of the no-trade interval is equivalent to super-additivity at certainty of the capacity and concavity of the utility function.

We further generalize Dow and Werlang's result by allowing for negative prices or equally we prove that the no-trade interval result is robust when considering assets which can yield non-positive outcomes.

We furthermore make precise the weak uncertainty aversion behavior of the agent, characteristic of the existence of such an interval, by proving that super-additivity at certainty of the capacity and concavity of the utility function are also equivalent to attraction by perfect hedging and preference for comonotone diversification.

Note that similar results had already been proved by Chateauneuf and Tallon

¹i.e. for all $A, B \in 2^S$ such that $A \cap B = \emptyset$, $p_{\mathcal{A}}(A \cup B) \geq p_{\mathcal{A}}(A) + p_{\mathcal{A}}(B)$

² v super-additive at certainty means that $v(A) + v(A^c) \leq 1$ for all event A

in [6] where they showed that for a CEU DM, preference for comonotone diversification is equivalent to the concavity of the utility function. Our proof that super-additivity at certainty of the capacity and concavity of the utility function implies attraction by perfect hedging and that, conversely, attraction by perfect hedging implies super-additivity at certainty of the capacity is inspired by a paper of Abouda and Chateauneuf [3] where, however, the context is that of risk for a RDEU agent.

Finally, we show that for a CEU DM endowed with an increasing utility function, the existence of the no-trade interval is equivalent both to aversion to some specific increase of uncertainty and to subjective increasing risk.

2 Definitions and notations

2.1 Elementary definitions of decision making under uncertainty

The distinction between risk (situations where there exists an objective probability distribution, known by the DM) and uncertainty (situations where there is no objective probability distribution, or it is unknown to the DM) is due to Knight [10].

Under (non-probabilized) uncertainty, a decision is a mapping, called act, from the set of states (of nature) Ω into the set of outcomes \mathbb{R} .

Exactly one state is the "true state", the other states are not true. A DM is uncertain about which state of nature is true and has not any influence on the truth of the states.

Let (Ω, \mathcal{A}) be a measurable space, \mathcal{B}_∞ be the set of \mathcal{A} -measurable bounded mappings from Ω to \mathbb{R} corresponding to all possible decisions and $X \in \mathcal{B}_\infty$.

We will denote \succsim the preference relation of the DM on the set of all acts \mathcal{B}_∞ .

For any pair of acts X, Y , $X \succsim Y$ will read X is (weakly) preferred to Y by the DM, $X \succ Y$ means that X is strictly preferred to Y , and $X \sim Y$ means that X and Y are considered as equivalent by the DM.

As usual, for all $X, Y \in \mathcal{B}_\infty$, we write: $X \succ Y$ if $X \succsim Y$ and not $Y \succsim X$; $X \lesssim Y$ if $Y \succsim X$; $X \prec Y$ if $Y \succ X$; $X \sim Y$ if $X \succsim Y$ and $Y \succsim X$.

Definition 2.1 A function $V : \mathcal{B}_\infty \rightarrow \mathbb{R}$ represents \succsim if

$$X \succsim Y \Leftrightarrow V(X) \geq V(Y) \text{ for all } X, Y \in \mathcal{B}_\infty.$$

2.2 The Choquet integral and CEU model

In the CEU model a certain class of set functions (capacities) is used in order

to represent the preference relation. We now define them and give some of their properties that will be used in the remainder of the article.

Definition 2.2 *v is a (normalized) capacity on (Ω, \mathcal{A}) if $v : \mathcal{A} \rightarrow [0, 1]$ is such that $v(\emptyset) = 0$, $v(\Omega) = 1$ and $\forall A, B \in \mathcal{A}, A \subset B \Rightarrow v(A) \leq v(B)$.*

Definition 2.3 *Let v be a capacity on (Ω, \mathcal{A}) , v is convex if $\forall A, B \in \mathcal{A}$, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$.*

Definition 2.4 *The Choquet integral of $X \in \mathcal{B}_\infty$ with respect to the capacity v is $I(X) = \int X dv$ where $\int X dv = \int_{-\infty}^0 (v(X \geq t) - 1) dt + \int_0^{+\infty} v(X \geq t) dt$.*

Definition 2.5 *$X, Y \in \mathcal{B}_\infty$ are comonotone if $\forall s, t \in \Omega$, $(X(s) - X(t))(Y(s) - Y(t)) \geq 0$ (i.e. X and Y vary in the same direction).*

Definition 2.6 *The core of a capacity v is defined by $C(v) = \{(finitely\ additive)\ probabilities\ P : P(A) \geq v(A) \forall A \in \mathcal{A}\}$.*

Definition 2.7 *We say that a DM satisfies the CEU model if his (her) preferences can be represented through an increasing utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ and a capacity v on (Ω, \mathcal{A}) which models his (her) personal evaluation of the likelihood of events. The representation of his (her) preferences is given by*

$$I \circ u : X \in \mathcal{B}_\infty \mapsto \int_{\Omega} (u \circ X) dv.$$

In the CEU model, preferences depend on the one hand on a utility function (which reflects the perception of wealth) and on the other hand on a capacity (reflecting the perception of the occurrence of events). This preferences representation is attractive for at least two reasons: it better represents real choices and allows for a separation between attitude towards uncertainty and attitude towards wealth.

Note that the two attitudes are mixed in the expected utility model, where they are both represented by the utility function.

Note also that, under risk, similar models are proposed by Kahneman and Tversky [9], Quiggin [12], and Yaari [19]. These models are known under the denomination of rank-dependent expected utility (RDEU).

3 Models and results

The study is focused on the case of uncertainty, that is on non-probabilized risk Choquet expected utility model (Schmeidler [15]). Preferences are then represented by the Choquet integral of a utility function u with respect to a capacity v .

3.1 The result of Dow and Werlang

We now present the model and result of Dow and Werlang [7].

They consider a measurable space (Ω, \mathcal{A}) , a convex capacity v on \mathcal{A} and a utility function u assumed to be C^2 and such that $u' > 0$ and $u'' \leq 0$.

Under these assumptions, they obtain the following result about the behavior of the "risk averse" or "risk neutral" agent under uncertainty aversion (i.e. v convex).

Theorem 3.1 (Dow and Werlang) *A risk averse (resp. risk neutral) investor with certain wealth $W > 0$, who is faced with an asset which yields a present value X per unit, whose price is $p > 0$ per unit, will buy the asset if $p < I(X)$ (resp. $p \leq I(X)$). He (she) will sell the asset if $p > -I(-X)$ (resp. $p \geq -I(-X)$).*

Remark 3.2 *Let us recall that in the EU model, it is well-known that the DM being "risk averse" (resp. "risk neutral") is equivalent to the utility function being concave (resp. affine).*

Theorem 3.1 is very intuitive and offers an appealing interpretation of the uncertainty aversion in terms of pessimism since, according to a well-known theorem of Schmeidler [14] which says that v is convex if and only if $C(v) \neq \emptyset$ and $I(X) = \min_{P \in C(v)} E_P[X]$, the agent views as possible the set of probabilities above the convex capacity v and will evaluate all assets X by $\min_{P \in C(v)} E_P[X]$.

So, when the price p is less than $I(X)$, the DM will buy a strictly positive amount of X since he (she) considers that the price is lower than the worst expected value of X .

Conversely, when the price p is greater than $-I(-X)$, the DM will sell short a strictly positive amount of X since he (she) considers that the price is greater than $\max_{P \in C(v)} E_P[X]$ i.e. than the best expected value of X .

Thus, he (she) will have no position on the asset X if and only if its price p is between $I(X)$ and $-I(-X)$.

The intuition behind this finding may be grasped in the following example given by Mukerji and Tallon [11]:

Consider an asset that pays off 1 in state L and 3 in state H and assume that the DM is of the CEU type with capacity $v(L) = 0.3$ and $v(H) = 0.4$ and linear utility function. The expected payoff (that is, the Choquet integral) of buying a unit of the risky asset is given by $I(X) = 1 + (3 - 1) \times v(H) = 1 + 2 \times 0.4 = 1.8$. The payoff from going short on a unit of the risky asset is $I(-X) = 2 \times (v(L) - 1) + 1 \times (0 - 1) = 2 \times (-0.7) - 1 = -2.4$. Hence, if the price of the asset X lies in the interval $[1.8; 2.4]$, then the investor would strictly prefer a zero position to either going short or buying.

3.2 Generalization and extension of the result of Dow and Werlang

Let (Ω, \mathcal{A}) be a measurable space such that \mathcal{A} contains at least one non-trivial event. Let \mathcal{B}_∞ be the set of bounded \mathcal{A} -measurable mappings from Ω to \mathbb{R} . We consider a CEU DM with a C^1 utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ such that $u' > 0$ and a capacity v on \mathcal{A} which is non trivial in the sense that there exists at least one event $A \in \mathcal{A}$ such that $0 < v(A) < 1$.

Remark 3.3 *Note that Dow and Werlang only considered assets for which natural reservation prices $I(X)$ and $-I(-X)$ are positive, and so they limit their study to the natural case where the price p by unit is positive; furthermore since they also assume that the investor is endowed with an initial deterministic positive wealth W , any infinitesimal buying transaction is feasible. In this paper, we show the robustness of Dow and Werlang's result when the above restrictions on X and W are relaxed. For instance in situation of insurance, an insurer would face the losses X of a potential insuree, and indeed since $X < 0$ such an insurer would agree to cover (a fraction of) the losses only if the price p of X is negative. In other terms, we aim at showing that Dow and Werlang's result would also allow to determine the minimum premium that an insurer would agree to receive from an insuree. Similarly, for a negative initial wealth, we can also address the question of determining whether the DM would buy the asset at a negative price or whether he (she) intends to sell short at a positive price. Consequently from now on, we will assume that the DM is endowed with an initial wealth not necessarily positive and an asset giving not necessarily positive monetary outcomes.*

In this paper, we will exclude the possibility for the DM to borrow any amount of money in order to avoid intricacies linked with the interest rates. The study of the situation where the DM is allowed to borrow is postponed to a further paper. Accordingly the trading situations which are excluded are those where the DM is endowed with a non-positive initial wealth and either the unit price p of X is positive, and he (she) intends to buy some shares of the asset or the unit price is negative and he (she) intends to sell short.

Any situation other than those where $W \leq 0$ and either $p > 0$ and the DM intends to buy some shares of the asset or $p < 0$ and the DM intends to take a short position will be called a situation of feasible trade.

We therefore obtain the following generalization of Theorem 3.1.

Theorem 3.4 *The two following assertions are equivalent:*

(a) *For any $X \in \mathcal{B}_\infty$, $I(X) \leq -I(-X)$. Furthermore, in any situation of feasible trade the DM has no position in the asset X on the range of prices $[I(X), -I(-X)]$, and he (she) buys a positive amount of the asset X at prices below $I(X)$, and holds a short position at prices higher than $-I(-X)$.*

(b) $\left\{ \begin{array}{l} (1) v(A) + v(A^c) \leq 1 \text{ for all } A \text{ in } \mathcal{A}. \\ (2) u \text{ is concave.} \end{array} \right.$

Note that in contrast with Theorem 3.1, we obtain an equivalence between the two conditions, which furthermore is valid under weaker conditions. Indeed, we only require that v be super-additive at certainty i.e. $v(A) + v(A^c) \leq 1$ for all A in \mathcal{A} instead of being convex and that u be C^1 instead of being C^2 , and this even when non-positive prices are considered.

As an illustration of Theorem 3.4, one notices that a CEU insurer satisfying condition (b) of Theorem 3.4 i.e. weak uncertainty averse in a sense which will be made precise below, will agree to insure a potential insuree faced with losses $X < 0$ if and only if the premium is greater than $-I(X)$.

We now make precise the weak uncertainty aversion behavior, characteristic of the existence of such an interval.

3.3 Uncertainty aversion behavior

Theorem 3.4 can be interpreted in terms of the attitude of the DM towards uncertainty. More precisely, we have the following theorem:

Theorem 3.5 *A CEU DM will exhibit the no-transaction interval in the sense of Theorem 3.4 (a) if and only if:*

(3) *He (she) is attracted by perfect hedging*
(i.e. $[X, Y \in \mathcal{B}_\infty, X \succsim Y, \alpha \in [0, 1], \alpha X + (1 - \alpha)Y = a1_\Omega, a \in \mathbb{R}] \Rightarrow a1_\Omega \succsim Y$).

and

(4) *He (she) exhibits preference for comonotone diversification*
(i.e. $X, Y \in \mathcal{B}_\infty, X$ and Y comonotone, $X \sim Y \Rightarrow \alpha X + (1 - \alpha)Y \succsim Y \forall \alpha \in [0, 1]$).

Note that at least three equivalent definitions of perfect hedging might be given (see for instance Proposition 3.6 in Abouda [1]).³

Remark 3.7 *1) Preference for perfect hedging means that if the DM can attain certainty by a convex combination of two assets, then he (she) prefers certainty to one of these assets, which is one of the mildest requirements for uncertainty aversion, so we can also call it attraction for certainty.*

3

Proposition 3.6 (Abouda [1]): *The following assertions are equivalent:*

- (i) $[X, Y \in \mathcal{B}_\infty, \alpha \in [0, 1], \alpha X + (1 - \alpha)Y = a1_\Omega, a \in \mathbb{R}] \Rightarrow a1_\Omega \succsim X$ or $a1_\Omega \succsim Y$.
- (ii) $[X, Y \in \mathcal{B}_\infty, X \succsim Y, \alpha \in [0, 1], \alpha X + (1 - \alpha)Y = a1_\Omega, a \in \mathbb{R}] \Rightarrow a1_\Omega \succsim Y$.
- (iii) $[X, Y \in \mathcal{B}_\infty, X \sim Y, \alpha \in [0, 1], \alpha X + (1 - \alpha)Y = a1_\Omega, a \in \mathbb{R}] \Rightarrow a1_\Omega \succsim Y$.

Note that the implications (i) \Rightarrow (ii) \Rightarrow (iii) are obvious while in order to prove that (iii) \Rightarrow (i), we use the natural fact that for all $X, Y \in \mathcal{B}_\infty, \lambda \geq 0$ if $X \succsim Y$ then $X + \lambda \succsim Y$.

2) *Comonotone diversification is nothing but convexity of preferences restricted to comonotone random variables (see Schmeidler [15]), it is therefore a kind of uncertainty aversion. Note that any hedging (in the sense of Wakker [18]) is prohibited in this diversification operation. This type of diversification turns out to be equivalent, in the CEU model, to the concavity of u .*

In [3], Abouda and Chateauneuf considered the problem of attraction by perfect hedging in the context of the RDEU model. In this model, the beliefs of the DM are represented by a distortion of probability $v = f \circ P$, where the distortion function $f : [0, 1] \rightarrow [0, 1]$ is assumed to be continuous, increasing and such that $f(0) = 0$ and $f(1) = 1$. The preferences of the DM are represented via a strictly increasing continuous utility function u : the DM weakly prefers X to Y if and only if $I(u(X)) \geq I(u(Y))$, where I is the Choquet integral. They show that if u is assumed to be concave and C^1 , then attraction by perfect hedging is equivalent to f satisfying $f(p) + f(1 - p) \leq 1$ for all $p \in [0, 1]$. Note that this property implies that v is super-additive at certainty. Since the class of capacities which are super-additive at certainty contains many interesting examples which are not distortions of probability, it seems desirable to extend the result of Abouda and Chateauneuf to that class. This is what we show in Theorem 3.5 where we prove that super-additivity at certainty is the only relevant property. Indeed, we show in that theorem that attraction by perfect hedging is equivalent to super-additivity at certainty of the capacity and concavity of the utility function.

3.4 Aversion to increasing uncertainty and subjective increasing risk

Definition 3.8 *A CEU DM is symmetrical monotone uncertainty averse (SMUA) if for all $X, Y \in \mathcal{B}_\infty$, $X \succsim_{sm} Y \Rightarrow X \succsim Y$ where $X \succsim_{sm} Y$ means that there exists $Z \in \mathcal{B}_\infty$, Z comonotone with X such that $I(Z) = I(-Z)$ and $Y = X + Z$.*

Y represents a monotone symmetrical increase of uncertainty in relation to X . So, a DM is symmetrical monotone uncertainty averse if he (she) doesn't like the monotone symmetrical increase of uncertainty i.e. if he (she) always prefers X to Y .

A similar notion of monotone symmetrical risk aversion was already defined by Abouda and Chateauneuf [3] for a RDEU agent.

A RDEU agent is said to be symmetrical monotone risk averse (SMRA) if for all $X, Y \in \mathcal{B}_\infty$, $X \succsim_{SM} Y \Rightarrow X \succsim Y$ where $X \succsim_{SM} Y$ means that there exists $Z \in \mathcal{B}_\infty$, Z comonotone with X such that $E(Z) = 0$, $Z =_d -Z$ and $Y =_d X + Z$ (where $X =_d Y$ means that X has the same probability distribution as Y).

To illustrate the symmetrical monotone risk order, take the example given in [2] where a typical symmetrical monotone increase in risk is obtained by adding to an asset Y a comonotone asset Z of probability law $L(Z) = (-\epsilon, p; 0, 1 - 2p; \epsilon, p)$ where $\epsilon > 0$ and $0 \leq p \leq \frac{1}{2}$.

Indeed, consider the following example:

Let X, Y be two assets with probability laws:

$L(X) = (3000, \frac{4}{9}; 5000, \frac{4}{9}; 6000, \frac{1}{9})$ and $L(Y) = (0, \frac{1}{9}; 2000, \frac{3}{9}; 5000, \frac{1}{9}, 6000, \frac{3}{9}, 9000, \frac{1}{9})$.

Therefore $X \succsim_{SM} Y$ since $Y =_d X + Z$ for some Z comonotone with X such that $Z =_d -Z$ and of probability law: $L(Z) = (-3000, \frac{1}{9}; -1000, \frac{3}{9}; 0, \frac{1}{9}, 1000, \frac{3}{9}, 3000, \frac{1}{9})$.

Y is obtained from X through two simple symmetrical spreads: $Y =_d X + Z_1 + Z_2$ where $L(Z_1) = (-2000, \frac{1}{9}; 0, \frac{7}{9}; 2000, \frac{1}{9})$ and $L(Z_2) = (-1000, \frac{4}{9}; 0, \frac{1}{9}; 1000, \frac{4}{9})$.

Note that in the previous definition, the condition $I(Z) = I(-Z)$ is equivalent to $E(Z) = 0$ when v is a probability measure.

We now introduce a notion of aversion to subjective increasing risk which is nothing else in the particular case of real outcomes than the one introduced by Ghirardato and Marinacci in [8] p. 877.

Definition 3.9 Let $A, B \in \mathcal{A}$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. We say that $Y = y_1 1_{B^c} + y_2 1_B$ is a binary subjective mean-preserving spread (SMPS) of the binary act $X = x_1 1_{A^c} + x_2 1_A$ if $y_1 \leq x_1, x_2 \leq y_2, v(A) = v(B)$ and $y_1(1 - v(B)) + y_2 v(B) = x_1(1 - v(A)) + x_2 v(A)$.

Definition 3.10 A CEU DM is averse to binary SMPSs if he (she) prefers a binary act to every one of its binary SMPSs.

Theorem 3.11 Let a CEU DM be endowed with a C^1 increasing utility mapping $u : \mathbb{R} \rightarrow \mathbb{R}$ and a non trivial capacity v on \mathcal{A} , then the two following assertions are equivalent:

- (1) $\left\{ \begin{array}{l} (a) \text{ The DM is SMUA.} \\ (b) \text{ The DM is averse to binary SMPS.} \end{array} \right.$
- (2) $\left\{ \begin{array}{l} (a) v(A) + v(A^c) \leq 1 \text{ for all } A \text{ in } \mathcal{A}. \\ (b) u \text{ is concave.} \end{array} \right.$

4 Concluding comments

After recalling the pioneering result of Dow and Werlang on the no-trade interval of a CEU DM, we generalize this result by allowing for negative prices while merely weakening convexity of the capacity into super-additivity at certainty. We also prove that a DM will exhibit this no-trade interval if and only if he (she) is attracted by perfect hedging and has preference for comonotone diversification or equivalently if he (she) presents some kind of uncertainty aversion. Finally, we show that for a CEU DM endowed with an increasing utility function, the existence of the no-trade interval is equivalent both to aversion to some specific

increase of uncertainty and to subjective increasing risk. We generalize the result of Dow and Werlang, who restricted themselves to the case of positive initial wealth and prices, by allowing for non-positivity. Our goal was achieved under the assumption that borrowing was excluded. We intend in a future study to examine the robustness of our results when this restriction is removed.

5 Appendix

We now give two technical lemmas which will be useful in the sequel:

Lemma 5.1 *Let $Y \in \mathcal{B}_\infty$, $u : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 increasing function and let $g : \alpha \in \mathbb{R} \rightarrow g(\alpha) = I(u(W + \alpha Y)) \quad \forall W \in \mathbb{R}$. Then, g is differentiable from the right at 0 and*

$$g'_+(0) = u'(W)I(Y).$$

Proof :

Let $\alpha > 0$, from the mean value theorem, there exists $t \in [0, 1]^\Omega$ such that

$$\frac{u(W + \alpha Y) - u(W)}{\alpha} = u'(W + t\alpha Y)Y.$$

It follows from positive homogeneity and constant additivity of the Choquet integral that

$$\frac{g(\alpha) - g(0)}{\alpha} = \int_{\Omega} u'(W + t\alpha Y)Y dv.$$

Note that $W + t\alpha Y$ converges to W in \mathcal{B}_∞ when α goes to zero. Hence since u' is continuous and Y is fixed, $u'(W + t\alpha Y)Y$ converges to $u'(W)Y$ in \mathcal{B}_∞ when $\alpha \downarrow 0$. Since I is norm-continuous on \mathcal{B}_∞ , $\int_{\Omega} u'(W + t\alpha Y)Y dv$ converges to $\int_{\Omega} u'(W)Y dv$ when $\alpha \downarrow 0$.

Now, since $u'(W) \geq 0$ and $\int_{\Omega} u'(W)Y dv = u'(W) \int_{\Omega} Y dv$,

$$g'_+(0) = u'(W)I(Y).$$

□

Lemma 5.2 *Let $W \in \mathbb{R}$, $Y \in \mathcal{B}_\infty$ and $u : \mathbb{R} \rightarrow \mathbb{R}$ an increasing C^1 concave utility function. If $\alpha \geq 0$ and $I(Y) \leq 0$ then $W + \alpha Y \lesssim W$.*

Proof :

Suppose that $\alpha > 0$ and $I(Y) \leq 0$. Then $\alpha I(Y) \leq 0$ and so $W + \alpha I(Y) \leq W$.

By comonotonicity, $I(W + \alpha Y) \leq W$ and since u is increasing,

$$u(I(W + \alpha Y)) \leq u(W).$$

Now, since u is concave and increasing, Jensen's inequality for capacities (proved by Asano [5] Theorem 4 p. 231) can be applied, giving:

$$I(u(W + \alpha Y)) \leq u(I(W + \alpha Y))$$

and so

$$I(u(W + \alpha Y)) \leq u(W) = I(u(W)) \text{ i.e. } W + \alpha Y \lesssim W.$$

□

Proof of Theorem 3.4:

First we show that (b) implies (a) :

★ Let us prove that (b) (1) implies that for all $X \in \mathcal{B}_\infty$, $I(X) \leq -I(-X)$.

Let $X \in \mathcal{B}_\infty$, then

$$I(X) = \int_{-\infty}^0 (v(X \geq t) - 1) dt + \int_0^{+\infty} v(X \geq t) dt.$$

Note that since $t \mapsto v(X \leq t)$ is non-decreasing the set of its discontinuities is at most countable.

Therefore, for $a < b$, $\int_a^b v(X \leq t) dt = \int_a^b v(X < t) dt$.

Thus,

$$\begin{aligned} I(-X) &= \int_{-\infty}^0 (v(X \leq -t) - 1) dt + \int_0^{+\infty} v(X \leq -t) dt \\ &= \int_{-\infty}^0 (v(X < -t) - 1) dt + \int_0^{+\infty} v(X < -t) dt \\ &= -\int_{+\infty}^0 (v(X < t) - 1) dt - \int_0^{-\infty} v(X < t) dt \\ &= \int_{-\infty}^0 v(X < t) dt + \int_0^{+\infty} (v(X < t) - 1) dt \end{aligned}$$

And so,

$$\begin{aligned} I(X) + I(-X) &= \int_{\mathbb{R}} (v(X < t) + v(X \geq t) - 1) dt \\ &\leq 0 \text{ (since by hypothesis } v(A) + v(A^c) \leq 1 \text{ for all } A \text{ in } \mathcal{A}) \end{aligned}$$

i.e. $I(X) \leq -I(-X)$.

★ We now turn to the no-trade interval result.

Indeed all the proofs below make sense only in the case of feasible trades, and consequently are valid as stated in Theorem 3.4, only in these cases.

Let us first prove that if $p \geq I(X)$, the investor is at least as well off not holding the asset, as buying any positive amount α .

- Note that for any p , buying any positive amount α of the asset leads to the uncertain future wealth $W(\alpha) = W + \alpha(X - p)$.

The formula is clearly true if $p = 0$.

If $p > 0$, buying a positive amount α of the asset at price p requires an amount of money $\alpha p > 0$, hence $W(\alpha) = W - \alpha p + \alpha X = W + \alpha(X - p)$.

If $p < 0$, buying αX yields a gain equal to $-\alpha p$, so $W(\alpha) = W - \alpha p + \alpha X$, hence $W(\alpha) = W + \alpha(X - p)$.

In order to show that the investor prefers not to buy, it is then enough to see that $W + \alpha(X - p) \lesssim W$ if $\alpha > 0$, but this results directly from Lemma 5.2 since $I(X - p) = I(X) - p \leq 0$.

Let us now prove that if $p \leq -I(-X)$, the investor is at least as well off not selling short the asset, as selling short any positive amount α .

- Note that for any p , selling short any positive amount α of the asset leads to the uncertain future wealth $\widetilde{W}(\alpha) = W + \alpha(-X + p)$.

The formula is clearly true if $p = 0$.

If $p > 0$, selling short αX yields a gain equal to αp , so $\widetilde{W}(\alpha) = W + \alpha p + \alpha(-X)$, hence $\widetilde{W}(\alpha) = W + \alpha(-X + p)$.

If $p < 0$, selling short a positive amount α of the asset at price p requires an amount of money $-\alpha p > 0$, hence $\widetilde{W}(\alpha) = W + \alpha p - \alpha X = W + \alpha(p - X)$.

In order to show that the investor prefers not to sell short, it is then enough to see that $W + \alpha(p - X) \lesssim W$ if $\alpha > 0$, but this results from Lemma 5.2, by setting $Y = p - X$; actually $I(Y) = p + I(-X) \leq 0$.

It remains to prove that if $p < I(X)$ the DM will hold a positive amount α of the asset and that if $p > -I(-X)$, he (she) will hold a short position $\alpha > 0$.

Assume now that $p < I(X)$. We only need to show that $W(\alpha) \succ W(0) = W$ for some $\alpha > 0$ or equally that $g(\alpha) = I(u(W + \alpha(X - p))) > g(0)$ for some $\alpha > 0$.

From Lemma 5.1, $g'_+(0) = u'(W)I(X - p)$ hence $g'_+(0) > 0$, which completes the proof.

Finally let $p > -I(-X)$. We only need to show that $\widetilde{W}(\alpha) \succ \widetilde{W}(0) = W$ for some $\alpha > 0$ or equally that $g(\alpha) = I(u(W + \alpha(p - X))) > g(0)$ for some $\alpha > 0$.

From Lemma 5.1, $g'_+(0) = u'(W)I(p - X)$ hence $g'_+(0) > 0$, which completes the proof.

We now prove that (a) implies (b) :

★ If $I(X) + I(-X) \leq 0$ for all X in \mathcal{B}_∞ then $v(A) + v(A^c) \leq 1$ for all A in \mathcal{A} .
Indeed:

Let $A \in \mathcal{A}$ and $X = 1_A$.

$$I(X) = \int_{\Omega} X dv = v(X = 1) = v(A).$$

$$I(-X) = -1 + v(X = 0) = -1 + v(A^c).$$

But since $I(X) + I(-X) \leq 0$ by hypothesis, $v(A) + v(A^c) \leq 1$.

★ u is concave. Indeed:

Let $A \in \mathcal{A}$ such that $0 < v(A) < 1$, $x, y \in \mathbb{R}$ such that $x < y$ and set $X = x1_{A^c} + y1_A$ and $t = v(A)$.

Then,

$$I(X) = x + (y - x)t = ty + (1 - t)x.$$

Let $W = p = I(X) = ty + (1 - t)x$ and $\alpha = 1$. Note that since $W = p$, we are in a situation of feasible trade, so that according to (a), $p = I(X)$ implies $W + \alpha(X - p) \lesssim W$

$$i.e. I(u(W + X - p)) \leq u(W)$$

and since $W = p = ty + (1 - t)x$,

$$I(u(X)) \leq u(ty + (1 - t)x).$$

On the other hand, since u is increasing, and therefore $u(x) < u(y)$,

$$I(u(X)) = u(x) + (u(y) - u(x))t.$$

Finally,

$$u(x) + (u(y) - u(x))t \leq u(ty + (1 - t)x)$$

$$i.e. tu(y) + (1 - t)u(x) \leq u(ty + (1 - t)x).$$

From this, we conclude that u is concave by a result due to Hardy, Littlewood and Polya (see Wakker [17]) which states that it is enough for a continuous function to satisfy the concavity inequality for one $t \in (0, 1)$ in order to be concave.

□

Proof of Theorem 3.5:

By Theorem 3.4, it is enough to prove that (b) is equivalent to (3) and (4).

Chateauneuf and Tallon [6] showed that for a CEU DM, preference for comonotone diversification is equivalent to the concavity of the utility function i.e. that conditions (2) and (4) are equivalent. The proof that (1) and (2) \Rightarrow (3) and (3) \Rightarrow (1) is inspired by a paper of Abouda and Chateauneuf [3] where a similar result is proved under risk for a RDEU agent.

★ First we show that (1) and (2) implies (3) (cf. Abouda and Chateauneuf [3] Theorem 3.8 (iii) \Rightarrow (v)):

Let $X, Y \in \mathcal{B}_\infty$, $X \succsim Y$, and $\alpha \in [0, 1]$ such that $\alpha X + (1 - \alpha)Y = a1_\Omega$, $a \in \mathbb{R}$.

We want to prove that $a1_\Omega \succsim Y$ i.e. $u(a) \geq I(u(Y))$.

Since u is concave and increasing, Jensen's inequality implies that

$$I(u(X)) \leq u(I(X))$$

and

$$I(u(Y)) \leq u(I(Y)).$$

Furthermore, since $X \succsim Y$, $I(u(Y)) \leq I(u(X))$.

Consequently, $I(u(Y)) \leq \min(u(I(X)), u(I(Y)))$ (*).

Now, if $I(X) \geq I(Y)$ (resp. $I(Y) \geq I(X)$) then $I(a1_\Omega) \geq I(Y)$ (resp. $I(a1_\Omega) \geq I(X)$).

Indeed: assume for instance that $I(X) \geq I(Y)$, then

$$\begin{aligned} I(Y) &= \alpha I(X) + (1 - \alpha)I(Y) \\ &\leq I(\alpha X) + I((1 - \alpha)Y) \text{ (since } I(X) \geq I(Y)) \\ &\leq -I(-\alpha X) + I((1 - \alpha)Y) \text{ (since } v(A) + v(A^c) \leq 1 \forall A \text{ implies } I(X) \leq -I(-X) \forall X) \\ &\leq -I((1 - \alpha)Y - a1_\Omega) + I((1 - \alpha)Y) \text{ (since } \alpha X + (1 - \alpha)Y = a1_\Omega) \\ &= I(a1_\Omega). \end{aligned}$$

Therefore we see that,

$$\min(I(X), I(Y)) \leq I(a1_\Omega) = a$$

and since u is increasing we conclude that

$$\min(u(I(X)), u(I(Y))) = u(\min(I(X), I(Y))) \leq u(a)$$

which together with (*) proves that

$$I(u(Y)) \leq u(a) \text{ i.e. } a1_\Omega \succsim Y.$$

★ We now show that (3) implies (1) (cf. Abouda and Chateauneuf [3] Theorem 3.8 (v) \Rightarrow (iii)):

By contradiction: Suppose that there exists A in \mathcal{A} such that $v(A) + v(A^c) > 1$ (note that this implies $v(A) > 0$ and $v(A^c) > 0$).

Let $a \in \mathbb{R}$, $\epsilon > 0$, and set

$$X_\epsilon = (a - \epsilon v(A^c))1_A + (a + \epsilon v(A))1_{A^c}$$

and

$$Y_\epsilon = (a + \epsilon v(A^c))1_A + (a - \epsilon v(A))1_{A^c}.$$

Note that

$$\frac{1}{2}X_\epsilon + \frac{1}{2}Y_\epsilon = a(1_A + 1_{A^c}) = a1_\Omega.$$

Since u is increasing, $u(a - \epsilon v(A^c)) \leq u(a + \epsilon v(A))$, so that

$$I(u(X_\epsilon)) = (u(a - \epsilon v(A^c))(1 - v(A^c)) + u(a + \epsilon v(A))v(A^c)).$$

Also,

$$I(u(a)1_\Omega) = u(a) = u(a)(1 - v(A^c)) + u(a)v(A^c).$$

Therefore,

$$I(u(X_\epsilon)) - I(u(a)1_\Omega) = (1 - v(A^c))(u(a - \epsilon v(A^c)) - u(a)) + v(A^c)(u(a + \epsilon v(A)) - u(a)).$$

Now, a Taylor expansion of order 1 gives:

$$u(a - \epsilon v(A^c)) - u(a) = -\epsilon v(A^c)u'(a) + o(\epsilon)$$

$$u(a + \epsilon v(A)) - u(a) = \epsilon v(A)u'(a) + o(\epsilon)$$

From this we get

$$\begin{aligned} I(u(X_\epsilon)) - I(u(a)1_\Omega) &= (1 - v(A^c))(-\epsilon v(A^c)u'(a)) + v(A^c)(\epsilon v(A)u'(a)) + o(\epsilon) \\ &= \epsilon(v(A^c)u'(a)(v(A) + v(A^c) - 1)) + o(\epsilon) \end{aligned}$$

But, since $v(A) + v(A^c) > 1$, $u'(a) > 0$ and $v(A^c) > 0$, there exists $\epsilon_1 > 0$ with

$$I(u(X_\epsilon)) - I(u(a)1_\Omega) > 0 \text{ if } \epsilon \in (0, \epsilon_1].$$

In the same way we obtain that

$$I(u(Y_\epsilon)) - I(u(a)1_\Omega) > 0 \text{ if } \epsilon \in (0, \epsilon_2] \text{ for some } \epsilon_2 > 0.$$

So that, taking $\epsilon = \min(\epsilon_1, \epsilon_2)$, we obtain $a1_\Omega \prec X_\epsilon$ and $a1_\Omega \prec Y_\epsilon$ which contradicts perfect hedging.

★ We now turn to the proof of (2) implies (4) (cf. Chateauneuf and Tallon [6] Theorem 3 (ii) \Rightarrow (i)):

Let $X, Y \in \mathcal{B}_\infty$ be comonotone and such that $X \sim Y$ and let $\alpha \in (0, 1)$.

We want to prove that $\alpha X + (1 - \alpha)Y \succeq Y$ i.e. $I(u(\alpha X + (1 - \alpha)Y)) \geq I(u(Y))$.

This is easy, indeed:

$$\begin{aligned}
I(u(Y)) &= \alpha I(u(X)) + (1 - \alpha) I(u(Y)) \\
&= \alpha I(u(X)) + (1 - \alpha) I(u(Y)) \text{ (since } X \sim Y) \\
&= I(\alpha u(X) + (1 - \alpha) u(Y)) \text{ (since } u \text{ is increasing and so preserves comonotony)} \\
&\leq I(u(\alpha X + (1 - \alpha) Y)) \text{ (since } u \text{ is concave.)}
\end{aligned}$$

★ As for the proof that (4) implies (2), the proof given on \mathbb{R}_{++} by Chateauneuf and Tallon for (i) \Rightarrow (ii) in Theorem 3 of [6] remains valid on \mathbb{R} . □

Proof of Theorem 3.11:

★ (2) implies (1) (a) :

Let $X, Y \in \mathcal{B}_\infty$ with $X \succsim_{sm} Y$, so that there exists $Z \in \mathcal{B}_\infty$ comonotone with X such that $I(Z) = I(-Z)$ and $Y = X + Z$.

We want to prove that $X \succsim Y$ i.e. $I(u(X)) \geq I(u(Y))$.

1. Suppose that for all $t \in \Omega$, $Z(t) \geq 0$.

If $Z(t) > 0$ then, since u is concave, we have

$$\frac{u(X(t) + Z(t)) - u(X(t))}{Z(t)} \leq u'(X(t)) \leq M := \sup_{x \in X(\Omega)} u'(x).$$

M is finite since u' is continuous and X bounded. Also $M \geq 0$ since $u' \geq 0$.

So, $u(X(t) + Z(t)) \leq u(X(t)) + MZ(t) \quad \forall t \in \Omega$ such that $Z(t) > 0$.

This inequality is also obviously true for t such that $Z(t) = 0$.

2. Suppose that for all $t \in \Omega$, $Z(t) \leq 0$.

If $Z(t) < 0$ then, since u is concave, we have

$$\frac{u(X(t)) - u(X(t) + Z(t))}{-Z(t)} \geq u'(X(t)) \geq m := \inf_{x \in X(\Omega)} u'(x).$$

For the same reason as in 1., $0 \leq m < +\infty$.

So, $u(X(t) + Z(t)) \leq u(X(t)) + mZ(t) \quad \forall t \in \Omega$ such that $Z(t) < 0$.

This inequality is also obviously true for t such that $Z(t) = 0$.

3. Suppose that there exist s and t in Ω such that $Z(s) < 0$ and $Z(t) > 0$.

Since Z is comonotone with X , $X(t) - X(s) \geq 0$.

Furthermore, since u is concave,

$$\frac{u(X(t) + Z(t)) - u(X(t))}{Z(t)} \leq \frac{u(X(s)) - u(X(s) + Z(s))}{-Z(s)} \quad (*)$$

Now, let $M' = \sup_F \left\{ \frac{u(X(t)+Z(t))-u(X(t))}{Z(t)} \right\}$ where $F = \{t \in \Omega \mid Z(t) > 0\}$.

Clearly $M' \geq 0$ (since u is increasing) and from $(*)$ we see that

$$\frac{u(X(t) + Z(t)) - u(X(t))}{Z(t)} \leq M' \leq \frac{u(X(s)) - u(X(s) + Z(s))}{-Z(s)}$$

for $Z(s) < 0$ and $Z(t) > 0$. So that

$$u(X(t) + Z(t)) \leq u(X(t)) + M'Z(t) \quad \forall t \mid Z(t) > 0$$

$$u(X(s) + Z(s)) \leq u(X(s)) + M'Z(s) \quad \forall s \mid Z(s) < 0.$$

Obviously, the same inequality is also true if $Z(s) = 0$ or $Z(t) = 0$.

Therefore, in all cases, there exists an $M \geq 0$ such that

$$u(X + Z) \leq u(X) + MZ.$$

Now, since $v(A) + v(A^c) \leq 1$ for all A in \mathcal{A} , $I(Z) + I(-Z) \leq 0$ (see Theorem 3.4) and since, by hypothesis, $I(Z) = I(-Z)$, $I(Z) \leq 0$.

On the other hand, since $Y = X + Z$ and $u(X)$ is comonotone with MZ ,

$$I(u(Y)) = I(u(X + Z)) \leq I(u(X) + MZ) = I(u(X)) + MI(Z)$$

Therefore, since $M \geq 0$ and $I(Z) \leq 0$,

$$I(u(Y)) \leq I(u(X)) \text{ i.e. } X \succsim Y.$$

★ (2) (b) implies (1) (b) :

Let $X = x_1 1_{A^c} + x_2 1_A$ and $Y = y_1 1_{B^c} + y_2 1_B$ a binary SMPS of X , i.e. such that $y_1 \leq x_1 \leq x_2 \leq y_2$, $v(A) = v(B)$ and $x_1(1 - v(A)) + x_2 v(A) = y_1(1 - v(B)) + y_2 v(B)$.

In order to show that the DM is averse to binary SMPSSs, we need to prove that $X \succsim Y$ i.e. that $d := I(u(X)) - I(u(Y))$ is non-negative.

From the expressions of $I(u(X))$ and $I(u(Y))$:

$$I(u(X)) = u(x_1)(1 - v(A)) + u(x_2)v(A)$$

and

$$I(u(Y)) = u(y_1)(1 - v(A)) + u(y_2)v(A),$$

we get

$$d = (u(x_1) - u(y_1))(1 - v(A)) - (u(y_2) - u(x_2))v(A).$$

We also have,

$$x_1(1 - v(A)) + x_2v(A) = y_1(1 - v(A)) + y_2v(A)$$

i.e.

$$(x_1 - y_1)(1 - v(A)) = (y_2 - x_2)v(A) \quad (*).$$

-If $v(A) = 0$, then by (*) $x_1 = y_1$ so that $d = 0$.

-If $v(A) = 1$, then by (*) $x_2 = y_2$ so that $d = 0$.

-If $v(A) \in (0, 1)$, then $(y_2 - x_2)(y_1 - x_1) \neq 0$ and

$$d = (u(x_1) - u(y_1))v(A) \left(\frac{y_2 - x_2}{x_1 - y_1} \right) - (u(y_2) - u(x_2))v(A).$$

Therefore, since u is concave,

$$\frac{d}{(y_2 - x_2)v(A)} = \frac{u(x_1) - u(y_1)}{x_1 - y_1} - \frac{u(y_2) - u(x_2)}{y_2 - x_2} \geq 0$$

and since $(y_2 - x_2)v(A) > 0$, we see that $d \geq 0$.

★ (1)(a) implies (2)(a) :

Let $x \in \mathbb{R}$, $\epsilon > 0$ and $A \in \mathcal{A}$.

If $v(A) = 1$ and $v(A^c) = 0$, there is nothing to prove.

Otherwise $1 + v(A^c) - v(A) > 0$ which we will assume from now on.

Let

$$Y_\epsilon = (x - \epsilon)1_A + \left(x + \frac{1 + v(A) - v(A^c)}{1 + v(A^c) - v(A)} \epsilon \right) 1_{A^c}$$

and

$$Z_\epsilon = -\epsilon 1_A + \frac{1 + v(A) - v(A^c)}{1 + v(A^c) - v(A)} \epsilon 1_{A^c}.$$

One readily checks that $Y_\epsilon = x + Z_\epsilon$ and $I(Z_\epsilon) = \epsilon \left(\frac{v(A^c) + v(A) - 1}{1 + v(A^c) - v(A)} \right) = I(-Z_\epsilon)$, which shows that $x 1_\Omega \succ_{sm} Y_\epsilon$.

Now, since the DM is SMUA, we conclude from this that $x1_\Omega \succsim Y_\epsilon$ (i.e. $I(u(x1_\Omega)) \geq I(u(Y_\epsilon))$). Together with the expression of $I(u(Y_\epsilon))$:

$$I(u(Y_\epsilon)) = u(x - \epsilon) + \left(u \left(x + \frac{1 + v(A) - v(A^c)}{1 + v(A^c) - v(A)} \epsilon \right) - u(x - \epsilon) \right) v(A^c)$$

this gives

$$\frac{u(x) - u(x - \epsilon)}{\epsilon} \geq \frac{v(A^c)}{\epsilon} \left(u \left(x + \frac{1 + v(A) - v(A^c)}{1 + v(A^c) - v(A)} \epsilon \right) - u(x - \epsilon) \right).$$

Performing a Taylor expansion we then obtain

$$u'(x) \geq \frac{2u'(x)v(A^c)}{1 + v(A^c) - v(A)}$$

and since $u'(x) > 0$ by hypothesis, we conclude that $v(A) + v(A^c) \leq 1$.

★ (1)(b) implies (2)(b) :

Let $A \in \mathcal{A}$ such that $0 < v(A) < 1$ and $x_1, x_2 \in \mathbb{R}$ such that $x_1 \leq x_2$. Setting $\alpha = x_1(1 - v(A)) + x_2v(A)$, one readily checks that $X := x_11_{A^c} + x_21_A$ is a binary SMPS of $Y := \alpha1_{A^c} + \alpha1_A$ and since the DM is averse to binary SMPSs, $Y \succsim X$, that is

$$u(x_1(1 - v(A)) + x_2v(A)) \geq u(x_1)(1 - v(A)) + u(x_2)v(A)$$

which, according to a result of Hardy, Littlewood and Polya (see Wakker [17]), shows that u is concave.

□

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